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INTERSECTION PROPERTIES OF BOXES. PART I: AN UPPER-BOUND THEOREM

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ABSTRACT

Let $\mathscr{P}$ be a family of n boxes in R^d (with edges parallel to the coordinate axes). For $k = 0, 1, 2, \ldots$, denote by $f_k(\mathscr{P})$ the number of subfamilies of $\mathscr{P}$ of size $k + 1$ with non-empty intersection. We show that if $f_r(\mathscr{P}) = 0$ for some $r \leqq n$, then

$$f_k(\mathscr{P}) \leqq f_k(n, d, r), \qquad k = 1, \ldots, r-1,$$

where the $f_k(n, d, r)$ are certain definite numbers defined by (3.4) below. The result is best possible for each k. For $k = 1$ it was conjectured by G. Kalai (Israel J. Math. **48** (1984), 161–174). As an application, we prove a 'fractional' Helly theorem for families of boxes in R^d.

1. Introduction

In this paper we are concerned with intersection properties of finite families of boxes in R^d. We establish what may be called an Upper-bound Theorem for such families (see Theorem 3.2 below). In particular, we prove a conjecture proposed in 1984 by Gil Kalai.

A *box* in R^d is a cartesian product of the form

$$I_1 \times \cdots \times I_d,$$

where for each $j = 1, \ldots, d$, I_j is a non-empty closed convex set (or *interval*) on the x_j-axis. (Here we refer to a cartesian coordinate system of R^d which is kept fixed throughout the paper.) In other words, a box is a convex parallelohedron, possibly unbounded, with edges parallel to the coordinate axes. The entire space R^d is a box, and so is every hyperplane parallel to one of the coordinate

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hyperplanes

$$H_j := \{(x_1, \ldots, x_d) \in R^d \mid x_j = 0\}, \qquad j = 1, \ldots, d.$$

Intersection properties of boxes (or rather of families of such) have been the subject of numerous investigations. We refer the interested reader to Section 10 in Hadwiger, Debrunner and Klee [10] and to the papers by Santaló [19], Rényi, Rényi and Surányi [16], Asplund and Grünbaum [3], Burling [5], Wegner [22], Roberts [17], Danzer and Grünbaum [6], and Gyárfás, Lehel and Tuza [9], to mention only a few.

Here we shall study intersection properties of a different kind (to be described in a moment). Our interest in these arose when we stumbled upon Conjecture 6.1 in Kalai [12]. The present paper is the outcome of our (successful) effort to prove and at the same time extend the assertion of Kalai's conjecture.

We start with some definitions and general remarks.

Let $\mathscr{P}$ be a finite family of boxes in R^d. (Actually, all families considered in this paper will be finite.) The *intersection graph* of $\mathscr{P}$, denoted $G(\mathscr{P})$, is the graph whose vertices are in one-to-one correspondence with the members of $\mathscr{P}$ and in which two vertices are joined by an edge only when the corresponding sets have a common point. It is well known that $G(\mathscr{P})$ completely determines the intersection pattern (or *nerve*) of $\mathscr{P}$. That is, $G(\mathscr{P})$ can be used to decide whether or not a given subfamily $\mathscr{Q}$ of $\mathscr{P}$ has non-empty intersection. In fact, projecting the boxes into the coordinate axes and applying Helly's theorem for the real line, one finds that $\mathscr{Q}$ has non-empty intersection if, and only if, each pairwise intersection of sets in $\mathscr{Q}$ is non-empty. Therefore we say that two families of boxes in R^d are of the same *intersection type* provided their intersection graphs are isomorphic.

Now, for $k = 0, 1, 2, \ldots$, let $f_k(\mathscr{P})$ denote the number of subfamilies of $\mathscr{P}$ of size $k+1$ with non-empty intersection. Equivalently, $f_k(\mathscr{P})$ is the number of complete subgraphs (or *cliques*) of $G(\mathscr{P})$ having $k+1$ vertices. In particular, $f_0(\mathscr{P})$ is the number of vertices and $f_1(\mathscr{P})$ is the number of edges of $G(\mathscr{P})$.

For the remainder of this paper we assume that n and r are two given integers satisfying $1 \leqq r \leqq n$.

Kalai's conjecture can be stated as follows (see [12], p. 173):

CONJECTURE. *Let $\mathscr{P}$ be a family of n boxes in R^d. Suppose that no $r+1$ members of $\mathscr{P}$ have a common point. Then*

$$(1.1)\quad f_1(\mathscr{P}) \leqq \begin{cases} t_r(n), & \text{if } r \leqq d, \\ t_d(n-r+d)+(n-r+d)(r-d)+\binom{r-d}{2}, & \text{if } r \geqq d, \end{cases}$$

where $t_r(n)$ denotes the maximum possible number of edges in a graph on n vertices containing no complete subgraph on $r+1$ vertices.

We abbreviate the right-hand side of (1.1) by $f_1(n, d, r)$. Thus it is claimed that if $f_0(\mathscr{P}) = n$ and $f_r(\mathscr{P}) = 0$, then $f_1(\mathscr{P}) \leqq f_1(n, d, r)$. Or, in graph-theoretic terms, if $G(\mathscr{P})$ has n vertices and *clique number* at most r, then $G(\mathscr{P})$ has at most $f_1(n, d, r)$ edges. (The clique number is the size of a largest clique.)

We need to say a few words about the function $t_r(n)$. The letter t stands for Turán who determined the exact value of $t_r(n)$ in 1941 (Turán [20], [21]). This is regarded as the first major result in extremal graph theory (see Bollobás [4], p. 292). Partition the number n into r almost equal parts $n_1, \ldots, n_r$, say.† Then

$$(1.2)\qquad t_r(n) = \sum_{\substack{i,j=1 \\ i<j}}^{n} n_i n_j = \binom{n}{2} - \sum_{i=1}^{n} \binom{n_i}{2}.$$

In other words, $t_r(n)$ is the second elementary symmetric function of $n_1, \ldots, n_r$. (For a generalization, see Section 3 below.) More explicitly, if we write $n = pr + q$, where p and q are integers satisfying $0 \leqq q < r$, then q of the above parts are equal to $p+1$ and $r-q$ are equal to p. This yields

$$(1.3)\qquad t_r(n) = \binom{n}{2} - r\binom{p}{2} - pq.$$

Turán (loc. cit.) has also shown that there exists, up to isomorphism, only one graph on n vertices having $t_r(n)$ edges and clique number at most r. This graph, usually denoted $T_r(n)$ and called the *Turán graph*, is the complete r-partite graph on n vertices whose vertex classes are as nearly equal as possible. This means that the vertex set of $T_r(n)$ splits into r subsets containing $n_1, \ldots, n_r$ vertices, respectively, with two vertices joined by an edge if, and

† Arranged in increasing order, these parts are

$$\left[\frac{n}{r}\right], \left[\frac{n+1}{r}\right], \ldots, \left[\frac{n+r-1}{r}\right],$$

where [.] denotes the greatest-integer function. (Cf. (3.2) below.)

only if, they belong to different subsets. The standard notation for this graph is $K(n_1, \ldots, n_r)$.

The first aim of the present paper is to prove Kalai's conjecture. This will be done in Section 2. Here we point out that the right-hand side of (1.1) cannot be replaced by a smaller number. In fact, Kalai (loc. cit.) describes a family of n boxes in R^d, $\mathscr{C} = \mathscr{C}(n, d, r)$, which satisfies $f_r(\mathscr{C}) = 0$ and $f_1(\mathscr{C}) = f_1(n, d, r)$. When $r \leqq d$, $\mathscr{C}(n, d, r)$ consists of n_i distinct translates of H_i for $i = 1, \ldots, r$, where the n_i are the parts used in (1.2) above. Clearly, $G(\mathscr{C}(n, d, r)) \cong T_r(n)$. When $r \geqq d$, partition the number $n - r + d$ into d almost equal parts $n'_1, \ldots, n'_d$, say. Then $\mathscr{C}(n, d, r)$ consists of n_i distinct translates of H_i for $i = 1, \ldots, d$, as well as $r - d$ copies of R^d. In this case, $G(\mathscr{C}(n, d, r)) \cong K(n'_1, \ldots, n'_d, 1, \ldots, 1)$.

The main result of this paper is presented in Section 3. There we shall prove that if $\mathscr{P}$ is a family of n boxes in R^d with $f_r(\mathscr{P}) = 0$, then for $k = 1, \ldots, r - 1$,

$$f_k(\mathscr{P}) \leqq f_k(\mathscr{C}(n, d, r)). \tag{1.4}$$

Notice that (1.4) generalizes (1.1) and is, by definition, best possible for all k. Explicit expressions for the numbers $f_k(\mathscr{C}(n, d, r))$ will be obtained in Section 3. We call (1.4) the Upper-bound Theorem (UBT) for families of boxes in R^d, by analogy with the well-known UBT for convex polytopes first proved by McMullen [15], or the more recent UBT for families of (arbitrary) convex sets due independently to Kalai [12] and the author [7]. (Compare Section 3 for further remarks.)

A family of boxes in R^d which attains the upper bound in (1.4) for each k is said to be *extremal*. One example is, of course, provided by Kalai's family $\mathscr{C}(n, d, r)$, but in general there are many other examples (distinguished by their intersection types). For a detailed study of the geometrical properties of such families, the reader is referred to the forthcoming Part II of the present paper. There we shall, in fact, completely characterize the intersection graphs of extremal families in R^d (up to isomorphism).

The final Section 4 of this paper is devoted to an application of the Upper-bound Theorem. We shall use the inequalities (1.4) to establish a 'fractional' Helly-type theorem for families of boxes in R^d. (Here 'fractional' is understood in the sense of Katchalski and Liu [14]; the precise definition will be given in Section 4.) We thereby extend and slightly sharpen an earlier 'fractional' result for boxes due to Katchalski [13].

2. Proof of Kalai's conjecture

In this section we shall establish Kalai's conjecture described in the Introduction. In other words, we shall prove assertion (1.1).

We begin with a useful definition. Let $\mathscr{P}$ be a family of boxes in R^d, and let Q be an arbitrary member of $\mathscr{P}$. We call Q *exposed* in $\mathscr{P}$ provided Q has a supporting hyperplane H, say, such that the following is true: H is parallel to some coordinate hyperplane H_j, and if $P \in \mathscr{P}$ and $P \cap H = \emptyset$, then P lies in the open half-space bounded by H whose complement contains Q. (We express this latter property by saying that P is *separated* from Q by H.) The pair (Q, H) is then also called exposed in $\mathscr{P}$. Strictly speaking, the above description applies to the case where Q is d-dimensional. By definition, any member of dimension less than d is automatically exposed in $\mathscr{P}$ and forms an exposed pair with any hyperplane containing it (and parallel to some H_j).

It is clear that unless the family $\mathscr{P}$ consists only of copies of R^d it has at least one exposed member. Just start with a suitable hyperplane 'at infinity' and let it sweep across $\mathscr{P}$, always keeping it parallel to some fixed H_j, until for the first time it is about to leave a member of $\mathscr{P}$. This member is then exposed in $\mathscr{P}$.

As simple as it might appear, the idea of using exposed pairs will be our main geometrical tool in what follows. For example, given such a pair (Q, H), if a member of $\mathscr{P}$ intersects Q, then it necessarily intersects $Q \cap H$. (For this it is essential that $\mathscr{P}$ consists of boxes.) Thus the family of all non-empty intersections $P \cap Q$, where P runs through $\mathscr{P} \setminus \{Q\}$, may be regarded as a family of boxes in $H \cong R^{d-1}$ (with edges parallel to the induced coordinate axes). This fact will enable us to proceed by induction on the dimension.

Having made these preliminary remarks, let us now turn to Kalai's conjecture. Recall that $f_1(n, d, r)$ is, by definition, the right-hand side of (1.1). We need to compare the values of $f_1(n, d, r)$ and $f_1(n-1, d, r)$, when $n > r$. Since it is readily verified that

$$t_r(n) - t_r(n-1) = \left[\frac{r-1}{r}\, n\right],$$

we find

$$(2.1) \qquad f_1(n, d, r) - f_1(n-1, d, r) = \begin{cases} \left[\dfrac{r-1}{r}\, n\right], & \text{if } r \leqq d, \\[2ex] \left[\dfrac{d-1}{d}\,(n-r)\right] + r - 1, & \text{if } r \geqq d. \end{cases}$$

(See Lemma 3.1 for a generalization.) Now we can prove

THEOREM 2.1. *Let $\mathcal{P}$ be a family of n boxes in R^d. Suppose that no $r+1$ members of $\mathcal{P}$ have a common point, i.e., $f_r(\mathcal{P})=0$. Then $f_1(\mathcal{P}) \leqq f_1(n,d,r)$, where $f_1(n,d,r)$ is defined by* (1.1) *above.*

PROOF. We use induction on d and n. Let $\mathcal{P}$ be a family of boxes as described in the theorem. Since $f_1(\mathcal{P}) \leqq t_r(n)$ is obvious from the definition of Turán's function, we may assume that $r \geqq d$. First consider the case $d=1$. Then $\mathcal{P}$ consists of intervals on the real line, and the assertion takes the form

$$f_1(\mathcal{P}) \leqq \binom{r}{2} + (r-1)(n-r).$$

This was proved by Abbott and Katchalski [1]. (Proofs also appear in [2], [7], and [12]; see the remarks preceding (3.1) below.) Next suppose $n=r$. In this case the assertion reads

$$f_1(\mathcal{P}) \leqq \binom{n}{2},$$

which is trivial. Finally, suppose that $d>1$ and $n>r$. In view of (2.1) it suffices to show that some member of $\mathcal{P}$ intersects at most

$$\left[\frac{d-1}{d}\,(n-r)\right] + r - 1$$

other members of $\mathcal{P}$. We claim that such a set can be found among the exposed members of $\mathcal{P}$. In fact, we shall prove a bit more:

(2.2) *There exists an exposed pair (Q,H) in $\mathcal{P}$ with the property that at least $[(n-r+d-1)/d]$ members of $\mathcal{P}$ are separated from Q by H.*

Notice that

$$\left[\frac{d-1}{d}\,(n-r)\right] + r - 1 + \left[\frac{n-r+d-1}{d}\right] = n-1,$$

so once (2.2) is proved, we are done. Notice, too, that no set in $\mathcal{P}$ can intersect more than $r-1$ other sets. Hence (2.2) is trivially true when $d=1$. Suppose, then, that $d>1$ and choose (Q,H) to be any exposed pair in $\mathcal{P}$. (Since $n>r$, such pairs exist.) Now (Q,H) either has the desired property or not. Assume it has not. Then

$$m \geqq \left[\frac{d-1}{d}(n-r)\right] + r + 1, \tag{2.3}$$

where m is the number of sets in

$$\mathcal{Q} := \{P \cap H \mid P \in \mathcal{P}, P \cap H \neq \varnothing\}.$$

We may regard $\mathcal{Q}$ as a family of boxes in R^{d-1}. Note that $m > r$. Hence by the induction hypothesis, $\mathcal{Q}$ has an exposed pair (Q_0, H_0), say, such that at least $[(m-r+d-2)/(d-1)]$ members of $\mathcal{Q}$ are separated from Q_0 by H_0. Now $Q_0 = Q' \cap H$ for some $Q' \in P$, and $H_0 \subset H$ is $(d-2)$-dimensional. Let H' be the hyperplane in R^d orthogonal to H which contains H_0. Going back to the original sets we deduce that at least $[(m-r+d-2)/(d-1)]$ members of $\mathcal{P}$ are separated from Q' by H'. Of course, Q' need not be exposed in $\mathcal{P}$. But in this case $\mathcal{P}$ has an exposed pair (Q'', H'') such that H'' is parallel to H' and lies on the same side of H' as Q' does. We claim that Q'' is the set (or one of the sets) we are looking for. In fact, all we have to do is to check that

$$\left[\frac{m-r+d-2}{d-1}\right] \geqq \left[\frac{n-r+d-1}{d}\right].$$

In view of (2.3), this will follow from

$$\left[\frac{\left[\frac{d-1}{d}(n-r)\right]}{d-1}\right] \geqq \left[\frac{n-r-1}{d}\right]. \tag{2.4}$$

Write $n-r = pd+q$, where p and q are integers satisfying $0 \leqq q < d$. Then (2.4) reduces to

$$\left[\frac{\left[\frac{d-1}{d}q\right]}{d-1}\right] \geqq \left[\frac{q-1}{d}\right].$$

In this inequality the left-hand side is always 0. The right-hand side is 0, when $q > 0$, and -1, when $q = 0$. This proves the theorem. □

As stated in the Introduction, the upper bound obtained in Theorem 2.1 is the best posible. Suppose it is attained by $\mathcal{P}$, that is, $f_1(\mathcal{P}) = f_1(n, d, r)$. Then, by (2.1), each member of $\mathcal{P}$ intersects at least

$$\left[\frac{d-1}{d}(n-r)\right]+r-1$$

other members, and some set in $\mathscr{P}$ (exposed, if $n > r$) intersects precisely that many members. For later use (in Part II) we record the following fact:

(2.5) *If $f_1(\mathscr{P}) = f_1(n,d,r)$ and, moreover, $n-r$ is divisible by d, then for each exposed pair (Q,H) in $\mathscr{P}$, Q intersects precisely*

$$\left[\frac{d-1}{d}(n-r)\right]+r-1$$

other members of $\mathscr{P}$ and is separated by H from the remaining $[(n-r+d-1)/d]$ members.

Assume, to the contrary, that Q intersects more than the above number of sets in $\mathscr{P}$. Since the inequality (2.4) is strict when d divides $n-r$, it follows that Q' (the set used in the proof of Theorem 2.1) intersects less than the above number of sets. This is a contradiction.

If $n-r$ is not a multiple of d, then assertion (2.5) need not hold. (It will be shown in Part II that (2.5) holds only when d divides $n-r$.) For example, take $r=3$ and consider the two families of 6 rectangles in the plane illustrated in Fig. 1. Both families attain the upper bound $f_1(6,2,3)=11$, and up to intersection type they are the only families with this property. The family on the left is (isomorphic to) Kalai's example $\mathscr{C}(6,2,3)$. Each family has 4 exposed members, two of which (the shaded ones) intersect 4, and not 3, other members.

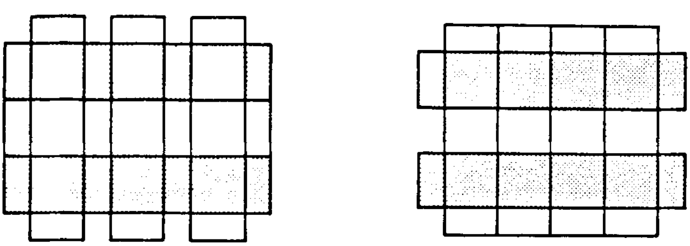

Fig. 1.

3. An upper-bound theorem

As before, we are concerned with families of n boxes in R^d in which no $r+1$ members have a common point. Given such a family $\mathscr{P}$, recall that $f_k(\mathscr{P})$ denotes the number of intersecting subfamilies of $\mathscr{P}$ of size $k+1$. Equivalently, $f_k(\mathscr{P})$ is the number of cliques of size $k+1$ in the intersection graph $G(\mathscr{P})$.

Having proved Kalai's conjecture, that is, having determined the best upper bound for $f_1(\mathscr{P})$ as a function of n, d and r, it is natural to inquire about best upper bounds for the numbers

$$f_2(\mathscr{P}), \ldots, f_{r-1}(\mathscr{P})$$

as well. It will be seen that after the bound on $f_1(\mathscr{P})$ has been established, the remaining bounds are obtained almost for free. No further geometry will be needed.

This section is devoted to the statement and proof of what we call the Upper-bound Theorem UBT for families of boxes in R^d. Namely we shall show that, subject to the above assumptions,

$$f_k(\mathscr{P}) \leqq f_k(\mathscr{C}(n,d,r)) \qquad \text{for } k = 1, \ldots, r-1,$$

where $\mathscr{C}(n,d,r)$ is Kalai's family described in Section 1. (Compare (1.4).) Furthermore, we shall prove that if $r \geqq d$ and equality holds in the above relation for some $k \in \{d, \ldots, r-1\}$, then equality holds for each $k \in \{1, \ldots, r-1\}$. The reader may observe that this behavior of the $f_k(\mathscr{P})$ is, in a sense, typical of upper-bound theorems. Consider, for instance, the UBT for families of (arbitrary) convex sets in R^d proved independently by Kalai [12] and the author [7]. (For a subsequent simplified proof, see Alon and Kalai [2].) This theorem asserts that if $r \geqq d$ and $\mathscr{K}$ is a family of n convex sets in R^d with $f_r(\mathscr{K}) = 0$, then

$$f_k(\mathscr{K}) \leqq \sum_{j=0}^{d} \binom{r-d}{k-j+1}\binom{n-r+d}{j} \tag{3.1}$$

for $k = 1, \ldots, r-1$. Moreover, equality in (3.1) for some $k \in \{d, \ldots, r-1\}$ forces equality for each $k \in \{1, \ldots, r-1\}$. The latter occurs, e.g., when $\mathscr{K}$ consists of $n-r+d$ hyperplanes in general position in R^d as well as $r-d$ copies of R^d.

To begin with, let us derive explicit expressions for the numbers

$f_k(\mathscr{C}(n,d,r))$. This is easily done with the help of symmetric functions. For $j = 0, 1, 2, \ldots$, set

$$s_j(n,r) := \sum \left[\frac{n+i_1}{r}\right] \cdots \left[\frac{n+i_j}{r}\right], \tag{3.2}$$

where the sum is extended over all systems of indices $i_1, \ldots, i_j$ such that $0 \leqq i_1 < \cdots < i_j \leqq r-1$. Thus $s_j(n,r)$ is the jth elementary symmetric function of the parts $n_1, \ldots, n_r$ used to represent $t_r(n)$ in Section 1. In particular, $s_0(n,r) = 1$ and $s_j(n,r) = 0$ for $j > r$. A more concise formula for $s_j(n,r)$ can be obtained by writing $n = pr + q$, where p and q are integers satisfying $0 \leqq q < r$. Then

$$s_j(n,r) = \sum_{i=0}^{j} \binom{q}{i}\binom{r-q}{j-i}(p+1)^i p^{j-i}. \tag{3.3}$$

Clearly, $s_2(n,r) = t_r(n)$. Hence (3.2) and (3.3) are generalizations of (1.2) and (1.3), respectively.

Now define, for $k = 0, 1, 2, \ldots$,

$$f_k(n,d,r) := \begin{cases} s_{k+1}(n,r), & \text{if } r \leqq d, \\ \displaystyle\sum_{j=0}^{d} \binom{r-d}{k-j+1} s_j(n-r+d,d), & \text{if } r \geqq d. \end{cases} \tag{3.4}$$

It is straightforward to check that $f_0(n,d,r) = n$ and $f_k(n,d,r) = 0$ fot $k \geqq r$, and that $f_1(n,d,r)$ is indeed equal to the right-hand side of (1.1), as it should be. More generally we have, for all k,

$$f_k(n,d,r) = f_k(\mathscr{C}(n,d,r)). \tag{3.5}$$

This follows at once from the definition of Kalai's family and the fact that if $\mathscr{P}$ arises from $\mathscr{P}'$ by adding $r-d$ copies of R^d, then for each k,

$$f_k(\mathscr{P}) = \sum_{j=0}^{d} \binom{r-d}{k-j+1} f_{j-1}(\mathscr{P}')$$

(with $f_{-1}(\mathscr{P}') := 1$).

Next we need to evaluate the difference between $f_k(n,d,r)$ and $f_k(n-1,d,r)$, when $n > r$. In doing this we shall make use of the identity

$$\left[\frac{\left[\frac{r-1}{r}\,n\right]+i}{r-1}\right]=\left[\frac{n+i}{r}\right], \tag{3.6}$$

which holds for $i=0,\ldots,r-2$ (and, of course, $r>1$). To prove (3.6), we may as well assume that $1\leqq n\leqq r$, since subtracting a multiple of r from n on both sides does not affect the equality. Hence

$$\left[\frac{r-1}{r}\,n\right]=n-1,$$

and we are left with showing that

$$\left[\frac{n+i-1}{r-1}\right]=\left[\frac{n+i}{r}\right]\qquad\text{for } i=0,\ldots,r-2.$$

This, in turn, is an immediate consequence of $n+i<2r-1$.

We can now state the crucial recursion formula which extends that of (2.1).

LEMMA 3.1. *For $n>r$ and $k=1,\ldots,r-1$, we have*

$$f_k(n,d,r)-f_k(n-1,d,r)$$
$$=\begin{cases} f_{k-1}\left(\left[\frac{r-1}{r}\,n\right],d-1,r-1\right), & \text{if } r\leqq d,\\ f_{k-1}\left(\left[\frac{d-1}{d}\,(n-r)\right]+r-1,d-1,r-1\right), & \text{if } r\geqq d.\end{cases}$$

PROOF. First suppose $r\leqq d$. Then it is to show that

$$s_{k+1}(n,r)-s_{k+1}(n-1,r)=s_k\left(\left[\frac{r-1}{r}\,n\right],r-1\right).$$

But to derive $s_j(n-1,r)$ from $s_j(n,r)$ is very simple. Just write $[(n-1)/r]$ in place of $[(n+r-1)/r]$ whenever the latter factor occurs on the right-hand side of (3.2). Since

$$\left[\frac{n+r-1}{r}\right]-\left[\frac{n-1}{r}\right]=1,$$

we easily deduce that

$$s_{k+1}(n,r)-s_{k+1}(n-1,r)=\sum\left[\frac{n+i_1}{r}\right]\cdots\left[\frac{n+i_k}{r}\right],$$

where the sum is over all $i_1,\ldots,i_k$ satisfying $0\leqq i_1<\cdots<i_k\leqq r-2$. In view of (3.6), this proves the assertion when $r\leqq d$.

Next we assume that $r\geqq d$. The reasoning in this case is completely analogous to that used above. According to (3.4), it is enough to show that

$$\begin{aligned}&s_j(n-r+d,d)-s_j(n-r+d-1,d)\\&=s_{j-1}\left(\left[\frac{d-1}{d}(n-r)\right]+d-1,d-1\right)\\&=s_{j-1}\left(\left[\frac{d-1}{d}(n-r+d)\right],d-1\right)\end{aligned}$$

for $j=1,\ldots,d$. This follows exactly as before, except that in (3.6) we have to replace n and r by $n-r+d$ and d, respectively. This finishes the proof of the lemma. □

Turning now to the main result of this paper (i.e., assertion (1.4)), we distinguish two cases.

First we remark upon the case $r\leqq d$. It should be clear by now that the problem of finding tight upper bounds for the $f_k(\mathscr{P})$ is, in this case, essentially a graph-theoretical problem. So, given a finite graph G, write $f_k(G)$ to denote the number of its cliques of size $k+1$. (Strictly speaking, $f_k(G)$ is already defined since Roberts [17] has shown that every graph is the intersection graph of some family of boxes.) Observe that $f_k(T_r(n))=s_{k+1}(n,r)$. Then we have what might be called an UBT for graphs:

(3.7) *Suppose G has n vertices and clique number at most r, i.e., $f_0(G)=n$ and $f_r(G)=0$. Then, for $k=1,\ldots,r-1$, $f_k(G)\leqq s_{k+1}(n,r)$. Moreover, if equality holds in this relation for some k, then $G\cong T_r(n)$, and equality holds for each k.*

When $k=1$, this becomes Turán's theorem. As far as we know, the above extension was first proved by Zykov [23] and later, independently, by Hadžiivanov [11] and Roman [18]. These proofs are rather long and somewhat involved. It seems that the simplest way to establish (3.7) is to imitate the proof of Theorem 3.2 below. (This approach is used in the author's note [8].)

For the remainder of this section we assume that $r\geqq d$.

Then, finally, we have

THEOREM 3.2. *Let $\mathscr{P}$ be a family of n boxes in R^d. Suppose that no $r+1$ members of $\mathscr{P}$ have a common point. Then, for $k = 1, \dots, r-1$,*

$$f_k(\mathscr{P}) \leqq f_k(n, d, r),$$

where $f_k(n, d, r)$ is defined by (3.4) *above. Moreover, if equality holds in this relation for some $k \in \{d, \dots, r-1\}$, then equality holds for each $k \in \{1, \dots, r-1\}$.*

Clearly, the first assertion generalizes Theorem 2.1. The reader will observe that Theorem 3.2 resembles the UBT for arbitrary convex sets (see (3.1) and the remarks following it) most closely in form.

PROOF. We use induction on d and n. Let $\mathscr{P}$ be a family of boxes as described in the theorem. Suppose first that $d = 1$. Then, by (3.4), the assertion takes the form

$$f_k(\mathscr{P}) \leqq \binom{r}{k+1} + \binom{r-1}{k}(n-r).$$

This is the special $d = 1$ of the inequality (3.1). For a proof of the equality assertion in this case, see [7]. Suppose next that $n = r$. Then for each j,

$$s_j(n - r + d, d) = \binom{d}{j}.$$

So the assertion reads

$$f_k(\mathscr{P}) \leqq \sum_{j=0}^{d} \binom{n-d}{k-j+1}\binom{d}{j} = \binom{n}{k+1},$$

which is trivial. If the bound $\binom{n}{k+1}$ is attained for some k, that is, if each $k+1$ members of $\mathscr{P}$ have a common point, then $\mathscr{P}$ has non-empty intersection, and the bound is attained for each k.

Suppose, finally, that $d > 1$ and $n > r$. Choose an exposed member of $\mathscr{P}$, say Q, which intersects at most

$$\left[\frac{d-1}{d}(n-r)\right] + r - 1$$

other members of $\mathscr{P}$. That such sets exist was demonstrated in the proof of Theorem 2.1. Set $\mathscr{P}' := \mathscr{P} \setminus \{Q\}$ and

$$\mathscr{P}'' := \{P \cap Q \mid P \in \mathscr{P}', P \cap Q \neq \emptyset\}.$$

Then $f_r(\mathscr{P}') = f_{r-1}(\mathscr{P}'') = 0$. Recall from Section 2 that $\mathscr{P}''$ can be viewed as a family of boxes in R^{d-1}. Therefore the induction hypothesis implies, for $k = 1, \ldots, r-1$,

$$f_k(\mathscr{P}') \leqq f_k(n-1, d, r), \tag{3.8}$$

$$f_{k-1}(\mathscr{P}'') \leqq f_{k-1}\left(\left[\frac{d-1}{d}(n-r)\right] + r - 1, d-1, r-1\right), \tag{3.9}$$

and from Lemma 3.1 we conclude that

$$f_k(\mathscr{P}) = f_k(\mathscr{P}') + f_{k-1}(\mathscr{P}'') \leqq f_k(n, d, r),$$

as required.

Suppose, now, that equality holds in the last relation for some $k \in \{d, \ldots, r-1\}$. Then equality must hold (for that particular k) in both (3.8) and (3.9). Appealing once more to the induction hypothesis we deduce that equality holds in (3.8) and (3.9) for each k. This in turn implies $f_k(\mathscr{P}) = f_k(n, d, r)$ for $k = 1, \ldots, r-1$, and the proof is complete. □

At this point we wish to repeat a remark made at the end of Section 1. In contrast to what is true for $r \leqq d$ (see (3.7) above), the families of boxes which attain the upper bounds in Theorem 3.2 are far from being unique. However, it is still possible to classify the intersection graphs of such families (called extremal in Section 1), when $r \geqq d$. This will be the topic of Part II of the present paper.

We conclude this section with an open problem. This, too, concerns the equality assertion of Theorem 3.2. We strongly believe that equality in $f_k(\mathscr{P}) \leqq f_k(n, d, r)$ for each $k = 1, \ldots, r-1$ is already implied by the fact that equality holds for some such k. At least we have no examples indicating that the stronger condition stated in the theorem is necessary for the result to hold. The proof of Theorem 3.2 shows that in order to establish what would be a strengthening of the UBT for boxes, it is enough to verify the following

CONJECTURE. *Let G be the intersection graph of a family of n boxes in R^d. Suppose that G has clique number at most r, where $d < r < n$, and $f_1(n, d, r)$ edges. Then G contains $f_2(n, d, r)$ triangles.*

4. A 'fractional' theorem

In an extension of the classical Helly intersection theorem, Katchalski and Liu [14] have introduced the concept of a 'fractional' Helly-type theorem. The most general result of this kind for families of convex sets was found by Kalai [12]. In this final section we establish the analogous 'fractional' result for families of boxes in R^d.

Briefly, the idea behind a 'fractional' theorem is as follows. Let ρ and α be positive real numbers such that the following is true: "If $\mathscr{K}$ is a family of n convex sets in R^d, and if the number of intersecting subfamilies of $\mathscr{K}$ of size $d+1$ exceeds $\alpha\binom{n}{d+1}$, then $\mathscr{K}$ contains a subfamily with non-empty intersection whose size exceeds ρn". Of course, this makes little sense when $\rho \geqq 1$ or $\alpha \geqq 1$. Katchalski and Liu [14] have shown that for any given $\rho < 1$, the number α in the above statement can indeed be chosen to satisfy $\alpha < 1$. (This is *a priori* not clear.) Furthermore, the smallest possible such α tends to 0 when ρ does. On the other hand, $\rho \to 1$ implies $\alpha \to 1$. This is essentially Helly's theorem.

Let $\alpha(\rho, d, k)$ be the smallest number α such that the statement above holds with $d+1$ replaced by some fixed $k \geqq d+1$. The problem is, of course, to explicitly determine $\alpha(\rho, d, k)$. This was done for $d=1$ and $k=2$ by Abbott and Katchalski [1], and for arbitrary d and k by Kalai [12]. Applying the inequalities (3.1) of the UBT for convex sets in R^d, Kalai proved

$$\alpha(\rho, d, k) = \sum_{j=0}^{d} \binom{k}{j} \rho^{k-j}(1-\rho)^j. \tag{4.1}$$

In particular, $\alpha(\rho, d, d+1) = 1-(1-\rho)^{d+1}$. See Section 4 in [12] for details and related results.

Here we shall obtain the analogue of assertion (4.1) for families of boxes in R^d. We begin by introducing the function $\eta(\rho, d, k, n)$ which plays the same role for boxes as does the function $\alpha(\rho, d, k, n)$ in [12] for arbitrary convex sets.

DEFINITION 4.1. *For* $0<\rho<1$ *and* $1 \leqq k < n$, *let* $\eta(\rho, d, k, n)$ *be the smallest real number* η *with the following property: If* $\mathscr{P}$ *is a family of* n *boxes in* R^d, *and if* $f_{k-1}(\mathscr{P}) > \eta\binom{n}{k}$, *then* $f_{[\rho n]}(\mathscr{P}) > 0$.

Clearly, $\eta(\rho, d, k, n)$ exists. It follows directly from the definition that $\eta(\rho, d, k, n)$ is increasing in ρ (for fixed d, k, n) and in d (for fixed ρ, k, n). For fixed ρ, d, and n, $\eta(\rho, d, k, n)$ is decreasing in k (see [12], p. 167).

Now for $0 < \rho < 1$ and $k > 1$, set

$$(4.2) \qquad \eta(\rho, d, k) := \sup_{n > k} \eta(\rho, d, k, n).$$

Thus $\eta(\rho, d, k)$ is the smallest number η such that the property described in Definition 4.1 holds for all families of boxes in R^d, regardless of how many members they have.

To evaluate $\eta(\rho, d, k)$, we shall make use of Theorem 3.2. As usual, $(d)_j$ denotes the falling factorial $d(d-1)\cdots(d-j+1)$ (with $(d)_0 := 1$).

Then we have

THEOREM 4.2. *For* $0 < \rho < 1$ *and* $k > 1$,

$$\eta(\rho, d, k) = \sum_{j=0}^{d} \binom{k}{j} \rho^{k-j}(1-\rho)^j \frac{(d)_j}{d^j}.$$

This should be compared with formula (4.1) above. Of course, the two expressions are the same for $d = 1$. Remember, however, that $\alpha(\rho, d, k)$ is defined only when $k > d$.

PROOF. Let ρ, d, and k be given, and suppose $n > k$. According to Theorem 3.2, $f_{k-1}(n, d, r)$ is the smallest integer such that, for any family $\mathscr{P}$ of n boxes in R^d, $f_{k-1}(\mathscr{P}) > f_{k-1}(n, d, r)$ implies $f_r(\mathscr{P}) > 0$. Since $f_{k-1}(n, d, r)$ is increasing in r (for fixed n, d, k), one easily obtains

$$\eta(\rho, d, k, n) = \binom{n}{k}^{-1} f_{k-1}(n, d, [\rho n]).$$

Now assume $[\rho n] \geqq d$. Then from (3.2) and (3.4) it follows that

$$\eta(\rho, d, k, n) = \binom{n}{k}^{-1} \sum_{j=0}^{d} \binom{[\rho n] - d}{k - j} s_j(n - [\rho n] + d, d),$$

where $s_j(n - [\rho n] + d, d)$ is the jth elementary symmetric function of the parts

$$n_i := \left[\frac{n - [\rho n] + d + i - 1}{d}\right], \qquad i = 1, \ldots, d.$$

We wish to compute $\lim_{n \to \infty} \eta(\rho, d, k, n)$. To do this we consider a typical term in the development of

$$(n)_k^{-1}([\rho n] - d)_{k-j} s_j(n - [\rho n] + d, d),$$

specified by j different parts $n_{i_0}, \ldots, n_{i_{j-1}}$, and split it into two factors (in a suitable way). As $n \to \infty$, these factors tend to easily calculated limits:

$$\lim_{n\to\infty} \prod_{l=0}^{k-j-1} \frac{[\rho n]-d-l}{n-j-l} = \rho^{k-j},$$

$$\lim_{n\to\infty} \prod_{l=0}^{j-1} \frac{n_{i_l}}{n-l} = \left(\frac{1-\rho}{d}\right)^j.$$

Now there are $\binom{d}{j}$ such terms for each j. Therefore, multiplying by $(k)_j$, we get

$$\lim_{n\to\infty} \eta(\rho, d, k, n) = \sum_{j=0}^{d} \binom{d}{j} (k)_j \rho^{k-j} \left(\frac{1-\rho}{d}\right)^j. \tag{4.3}$$

In order to show that the right-hand side of (4.3) is equal to $\eta(\rho, d, k)$, as claimed, it is enough to establish

$$\eta(\rho, d, k, 2n) \geqq \eta(\rho, d, k, n), \qquad \text{if } n > k. \tag{4.4}$$

Notice that we have dropped the assumption $[\rho n] \geqq d$ here.

The proof of (4.4) uses an idea of Kalai [12]. It does not really depend on the particular type of sets we are considering. Indeed, let $\mathscr{P}$ be any family of n sets, and define $2\mathscr{P}$ to consist of two copies of each member of $\mathscr{P}$. A purely combinatorial argument yields

$$\binom{2n}{k}^{-1} f_{k-1}(2\mathscr{P}) \geqq \binom{n}{k}^{-1} f_{k-1}(\mathscr{P})$$

(see [12], p. 168). We now choose $\mathscr{P}$ to be Kalai's family $\mathscr{C}(n, d, [\rho n])$ defined in Section 1. Then

$$\binom{n}{k}^{-1} f_{k-1}(\mathscr{P}) = \eta(\rho, d, k, n) \quad \text{and} \quad f_{[\rho n]}(\mathscr{P}) = 0,$$

which implies $f_{[2\rho n]}(2\mathscr{P}) = 0$. Hence

$$\binom{2n}{k}^{-1} f_{k-1}(2\mathscr{P}) \leqq \eta(\rho, d, k, 2n).$$

This establishes (4.4) and proves the theorem. □

The most interesting 'fractional' Helly-type theorem is obtained when $k = 2$. Then

$$\eta(\rho, d, 2) = 1 - \frac{(1-\rho)^2}{d}.$$

This is essentially due to Katchalski [13]. In other words, if the intersection graph of a family of boxes in R^d has more than

$$\left(1 - \frac{(1-\rho)^2}{d}\right)\binom{n}{2}$$

edges, where n is the number of its vertices, then it contains a clique of size $[\rho n] + 1$. The result is best possible in the sense described in Definition 4.1.

To conclude, let us mention three simple consequences of Theorem 4.2. First, the behavior of $\eta(\rho, d, k)$ for large d is given by

$$\eta(\rho, d, k) = 1 - \binom{k}{2}\frac{(1-\rho)^2}{d} + O(d^{-2}). \tag{4.5}$$

This is easily seen by writing $\eta(\rho, d, k)$ as a polynomial in $1/d$. Next we have

$$\lim_{\rho \to 1} \eta(\rho, d, k) = 1, \tag{4.6}$$

and finally, perhaps more interesting,

$$\lim_{\rho \to 0} \eta(\rho, d, k) = \frac{(d)_k}{d^k}. \tag{4.7}$$

In particular, $\eta(\rho, d, k)$ tends to 0 (as $\rho \to 0$) iff $k > d$. (Compare the remark following Theorem C in Katchalski [13].)

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